# An $\mathcal{O}(n\sqrt{n}\log\log n)$ average case algorithm for the maximum induced matching problem in permutation graphs

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***Abstract*—Let $G = (V, E)$ be an undirected graph, where $V$ is the vertex set and $E$ is the edge set. A subset $M$ of $E$ is an *induced matching* of $G$ if $M$ is a matching of $G$ and no two edges in $M$ are joined by an edge. Finding a *maximum induced matching* is a $\mathbb{NP}$-Hard problem on general graphs, even on bipartite graphs. However, this problem can be solved in polynomial time in some special graph classes such as weakly chordal, chordal, interval and circular-arc graphs. In this paper, we introduce a maximum induced matching algorithm in permutation graphs with $\mathcal{O}(|V|k(G)\log\log(|V|))$ time in worst case complexity and $\mathcal{O}(|V|\sqrt{|V|}\log\log(|V|))$ time in average case complexity, where $k(G)$ is the cardinality of the minimum clique cover set. The approach is to reduce the size of vertex set of $L(G)^2$ without changing the cardinality of its maximum independent set. Our algorithm has better time complexity than the best known algorithm in both worst case and average case.***

## I. Introduction

For a finite set $V = 1, 2, \ldots, n$ and a permutation $\pi = [\pi[1], \pi[2], \ldots, \pi[n]]$ of $V$, let $G(\pi) = (V, E)$ denote the undirected graph satisfying $(i, j) \in E$ iff $(i - j)(\pi^{-1}[i] - \pi^{-1}[j]) < 0$, where $\pi^{-1}$ is the reverse permutation of $\pi$. A graph $G$ is a *permutation* graph iff there exists a permutation $\pi$ such that $G = G(\pi)$.

A permutation graph can be visualized through its *permutation diagram.* A permutation diagram of a permutation graph $G(\pi)$ consists of two parallel horizontal channels, called top channel and bottom channel. In the top channel, there are $n$ points labeled $1, 2, \ldots, n$ from left to right. We put the numbers $\pi[1], \pi[2], \ldots, \pi[n]$ with the same order in the bottom channel. For each $i$, draw a straight line, labeled $i$, joining two numbers $i$ in the top channel and in the bottom channel. The line $i$ intersects the line $j$ iff there is an edge between $i$ and $j$ in $G(\pi)$. From here we use the word "line" as a reference to its corresponding vertex. Figure 1 shows a permutation diagram.

An edge set $M \subset E$ is called a matching of $G$ iff there does not exist a pair of edges in $M$ with a common vertex. An induced matching of $G$ is a matching where the distance between two arbitrary vertices in two different edges is at least two. Induced matchings have many applications in practice, such as for ensuring the security of the information transferring channel, VLSI and network flow problems. Recently, these problems attracted much attention because of their theoretical interest and practical motivation, see some related papers [2], [6], [7], [9], [11], [12], [13], [18], [19], [21], [22].

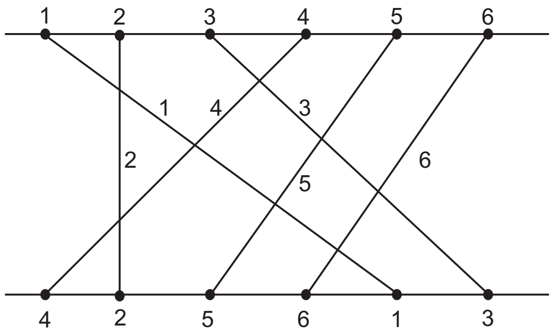


Fig. 1. Permutation diagram

A subset $S$ of $V$ is called an independent set if no two vertices in $S$ are adjacent. It is well-known that finding a maximum independent set (MIS) of a graph is a $\mathbb{NP}$-Hard problem. The number of vertices in a MIS of $G$ is called the independence number, denoted by $\alpha(G)$.

The maximum induced matching (MIM for short) problem is first proposed in 1989 by Cameron [3]. While the maximum matching problem can be solved in polynomial time in an arbitrary graph, the MIM problem is a $\mathbb{NP}$-Hard problem, even for bipartite graphs. The MIM problem on general graphs can be solved by some brand-and-reduce algorithms, but all of them have exponential time complexity. Chang et al. introduced an improvement for the previous brand-and-reduce algorithm and obtained an $\mathcal{O}(1.4321^n)$ time algorithm [10]. Recently, Xiao et al. proposed two algorithms, one of them can solve MIS problem in $\mathcal{O}(1.4213^n)$ times and polynomial space, the other is an $\mathcal{O}(1.3752^n)$ times and exponential space algorithm [24].

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However, the MIM problem can be solved in polynomial time for interval graphs and chordal graphs [3], Circular arc graphs [15], Trapezoid graphs and co-comparability graphs [16], Asteroidal-triple-free graphs [4] [8], Weakly chordal graphs [5], Interval-filament graphs [4].

$G^k = (V, E')$ is defined as a graph having the same vertex set with $G$. Two vertices $u, v$ in $G^k$ are adjacent iff there exists a path from $u$ to $v$ of length less than or equal to $k$.

Let $L(G)$ denote the line graph of $G$, *i.e.*, each edge of $G$ is a vertex of $L(G)$, two vertices of $L(G)$ are adjacent iff two corresponding edges share a common endpoint. Figure 2 shows an example of a graph $G$ and its line graph $L(G)$. An induced matching of a graph $G$ corresponds with an independent set of $L(G)^2$. So there will be a polynomial complexity algorithm for MIM whenever MIS of a graph can be found in polynomial time. In some circumstances, avoiding constructing fully the graph $L(G)^2$ may lead to better time complexity. There is algorithm that can find MIM in linear time in some special graphs including chordal graphs [1], interval graphs [16], tree graphs [25], [16] and permutation bipartite graphs [8].

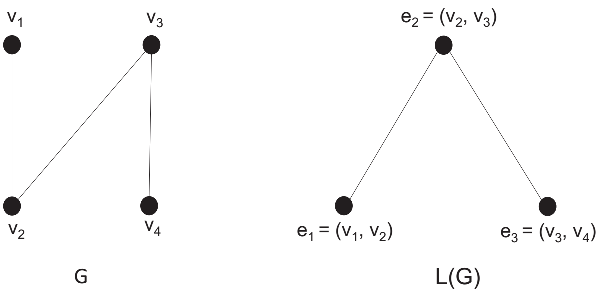


Fig. 2. Constructing the line graph

Permutation graphs have been proven to be co-comparability [17]. Golumbic et al. have introduced an $\mathcal{O}(|E|^2)$ algorithm finding MIM in co-comparability graph [16].

In this paper, we present an algorithm solving MIM problem with complexity $\mathcal{O}(n^2 \log(n))$ in worst case and $\mathcal{O}(n\sqrt{n}\log(n))$ in average case. Our approach is to reduce the size of the graph $L(G)^2$ but the cardinality of its MIS is still unchanged. We find some $\alpha$-redundant vertices which their removal do not decrease the independence number of $L(G)^2$. Our algorithm has better time complexity than the best known algorithm, which is an $\mathcal{O}(|E|^2)$ algorithm [16], in both worst case and average case. And there is a high probability that the complexity of our algorithm is lower than $cn\sqrt{n}\log\log(n)$ with $c$ is a constant.

## II. Basic Notations

A subset $C$ of $V$ is a clique if it induce a complete graph from $G$. The number of vertices in a maximum clique of $G$ is $\omega(G)$. A clique cover of a graph is a partition $V = C_1 + C_2 + \ldots + C_n$ such that $C_i$ is a clique for every $i$. Let $k(G)$ be the cardinality of the minimum clique cover of graph $G$.

For every line in $G$, we define the relation "completely on the left", denoting by "$||$", which $u||v$ iff $u < v$ and $\pi^{-1}[u] < \pi^{-1}[v]$.

We use the same notation $(u, v)$ to imply a vertex of $L(G)$ or $L(G)^2$ and a trapezoid which has two diagonal lines $u$ and $v$.

## III. $L^2(G)$ of a Permutation Graph

**Lemma 1.** *$G = (V, E)$ is the permutation graph corresponding with a permutation $\pi$. Then $L^2(G)$ is a trapezoid graph.*

*Proof.* We can represent $G$ in a permutation diagram as in figure 1. We will prove that each $L^2(G)$ is isomorphic with the trapezoid graph $T$ in which vertices are trapezoid with 2 diagonal lines is 2 intersected lines of the $G$. Each vertex of $T$ matches with a vertex of $L^2(G)$. It is obvious that two trapezoids in $T$ intersect iff at least one of their diagonal line intersects the others. If there is an edge between two vertices $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ in $L^2(G)$, so one line in $\{u_1, v_1\}$ coincides or intersects with a line in $\{u_2, v_2\}$. Thus their corresponding trapezoids intersect each other. A similar proof can be used for solving the backward. □

We use the term trapezoid for vertices of $L^2(G)$ and any trapezoid graph.

## IV. The Construction of $E^*$

If we construct the $L^2(G)$ then the size of the input for finding MIS will be $\mathcal{O}(n^2)$. For example if the permutation is $[n, n-1, \ldots, 1]$, then the corresponding graph become complete, and the cardinality of $L^2(G)$ will be $n(n-1)/2$. Finding a MIS in this graph costs $\mathcal{O}(n^2 \log\log(n))$ time. We improve the algorithm by cutting down the size the of the vertex set in $L^2(G)$. Let denote this new (trapezoid) graph by $E^*$. For each line $u$ and a clique $C_i$, we only keep the line $v$ which is the minimum line in $C_i$ and satisfies $v > u$ and $\pi^{-1}[v] < \pi^{-1}[u]$. In different words, $v$ is the smallest line in $C_i$, which is larger than $u$ and cuts $u$. The vertex of new trapezoid $(u, v)$ will be added to $E^*$. In the graph in figure 3, we delete the vertex $(u, w)$ and only keep the vertex $(u, v)$.

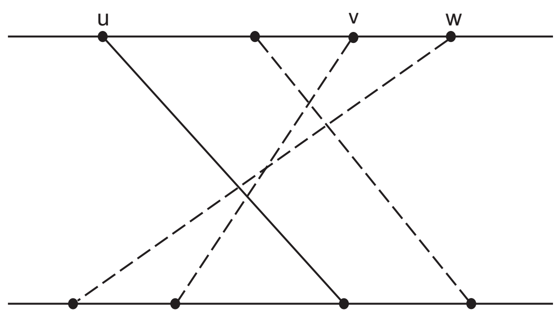


Fig. 3. The $\alpha$-redundant trapezoid

We will show that all trapezoids building from $u$ and a line in $C_i$ are $\alpha$-redundant except the chosen trapezoid.

**Lemma 2.** *If any arbitrary trapezoid does not intersect with a trapezoid with the diagonal lines $u$ and $w$ with $u < w$ and $w \in C_i$, it will also not cut the trapezoid $(u, v)$.*

*Proof.* Assume that a trapezoid $(x, y)$ does not intersect with $(u, w)$ in $L^2(G)$, we will prove $(x, y)$ does not intersects $(u, v)$, too. We have proven that two vertices in trapezoid diagram are adjacent iff there exists a pair of their diagonal lines cutting each other. Without losing the generality, we will prove that $x$ does not cut $v$, because $x$ and $u$ share no point. Hence $x$ cannot line in the segment $[u, w]$.
The first case is $x||w$. So $x < v$ because $v > u$. As $v$ is the minimum line cutting $u$ in $C_i$, so $\pi^{-1}[v]$ is the largest among the revert permutation of every member cutting $u$ in $C_i$, so $\pi^{-1}[x] < \pi^{-1}[v]$. Hence, $x$ is completely on the left of $v$.
The $w||x$ case can be solved by similar way. □

## V. MIM of Permutation Graphs

We describe our algorithm for finding a MIM of a permutation graph as below:

**Algorithm 1** Maximum induced matching in permutation graphs

1: **Input:** Permutation $\pi$
2: **Output:** A maximum induced matching of $G(\pi)$
3: Finding Clique Cover $CC(\pi) = (C_1, C_2, \ldots, C_q)$ of $G(\pi)$
4: $E^* = \emptyset$.
5: **for** each line $n$ in $G(\pi)$ **do**
6:     **for** each clique $C_i$ in $CC(\pi)$ **do**
7:         Find the smallest line $m$ in $C_i$ that $m < n$ and $\pi^{-1}[m] > \pi^{-1}[n]$
8:         $E^* = E^* \cup \{(m, n)\}$
9:     **end for**
10: **end for**
11: Find maximum independent set $S$ of $E^*$
12: Return $M = \{(i, j)|(i, j) \in S\}$

Step 3, we can construct an algorithm computing $CC(\pi)$ in $\mathcal{O}(n \log\log(n))$ time by the following steps. We consider the list $\pi$ from the left to the right. At position $j$, if the list $C_i$ is empty, we immediately append the number $\pi[j]$ to the back of the list $C_i$. If $C_i$ is not empty, we find the first next line which is smaller than the last element in $C_i$. After completing $C_i$, we continue this procedure to find $C_{i+1}$. Y. Liang and C. Rhee [20] have proved the following lemma.

**Lemma 3.** *Constructing $CC(\pi)$ can be done in $\mathcal{O}(n \log\log(n))$.*

## VI. Correctness

Because an independent set of $E^*$ corresponds to an independent set of $L^2(G)$, so $\alpha(E^*) \leq \alpha(L^2(G))$.

Assume that $\mathcal{O} = \{e_1, e_2, \ldots e_m\}$ is an independent set of $L^2(G)$. We will prove that there exist an independent set $IS = \{e'_1, e'_2, \ldots, e'_m\}$ in $E^*$.

We will construct $IS$ from $\mathcal{O}$. At the initial step, $IS$ is assigned by $\{e_1, e_2, \ldots, e_m\}$, which mean $e'_j = e_j$ for every $j$ . Suppose that $e'_j \in E^*$ with every $j \leq k$. Let $e'_{k+1} = e_{k+1} = (u, w)$ and $w$ is in a clique $C_i$. We will prove that $e_{k+1}$ can be replaced by a vertex of $E^*$ denoted by $(u, v)$ with $v$ is the smallest line in $C_i$ which is larger than $u$ and cuts $u$.

As our proof, $(u, v)$ does not intersect any other trapezoid in $IS$. For all diagonal lines $l$ of a trapezoid $j$, the following conditions hold:

$$l||u \text{ and } l||v \text{ if } j < k+1$$
$$u||l \text{ and } v||l \text{ if } j > k+1$$

So we can replace $(u, w)$ by $(u, v)$, and this new trapezoid does not intersect any other trapezoids in $IS$. By mathematical induction we can build an independent set $IS$ from an arbitrary independent set $\mathcal{O}$ of $L^2(G)$ with the cardinality is $m$ and its elements are vertices of $E^*$. As a result, $\alpha(L^2(G)) \leq \alpha(E^*)$. And because $\alpha(E^*)$ does not exceed $\alpha(L^2(G))$, we conclude $\alpha(L^2(G)) = \alpha(E^*)$

## VII. Complexity

The number of elements in $E^*$ is $\mathcal{O}(nk(G))$, maximum independent set of $E^*$ can be found in $\mathcal{O}(|E| \log\log(|E|))$ [14]. So the time complexity of the algorithm is $\mathcal{O}(nk(G) \log\log(n))$. Evaluation for $k(G)$ in the worst case is $\mathcal{O}(n)$. So the algorithm have $\mathcal{O}(n^2 \log\log(n))$ time complexity.
In the average time case, the complexity is lower. We suppose that the distribution of the permutation $\pi$ is uniform. Subsequently, the distribution of $\pi^{-1}$ is uniform.

**Theorem 1.** *The average case complexity of the algorithm is $\mathcal{O}(n\sqrt{n} \log\log(n))$.*

To begin with the proof of this theorem, we consider the following lemma.

**Lemma 4.** *$k(G)$ is equal to the number of maximum increasing subsequence of $\pi^{-1}$.*

*Proof.* As $G$ is a perfect graph, so $k(G) = \alpha(G)$. An independent set of $G$ can be presented as a set $IS = v_1, v_2, \ldots, v_k$ in increase order. Because $v_i$ does not intersect $v_j$ $(v_i < v_j)$, then $\pi^{-1}[v_i] < \pi^{-1}[v_j]$. We get the result that the sequence $\pi^{-1}[v_1], \pi^{-1}[v_2], \ldots, \pi^{-1}[v_k]$ is an increasing subsequence of $\pi^{-1}$. So the number of members in MIS is precisely equal to the number of the maximum increasing subsequence of $\pi^{-1}$. □

**Lemma 5.** *Average number of $k(G)$ over all n-vertices permutation graphs is $\mathcal{O}(\sqrt{n})$.*

The following proof is based on the proof of D. Romik of the problem finding the longest increasing subsequence of a sequence [23].

*Proof.* We call $G_i$ as the subgraph of $G$ which has vertices from $1$ to $i$ and all of the edge between them and $S_G$ is the set of all permutation graphs which have $n$ vertices. We use $k_i$ and $\omega_i$ as symbol for the maximum independent set and maximum clique contain vertex $i$ of the graph $G_i$. It is obvious that $n$ pairs $(k_i, \omega_i)$ is distinguished since $k_j > k_i$ if $\pi^{-1}[j] > \pi^{-1}[i]$ and $\omega_j > \omega_i$ in other case. As the number of pairs $(k_i, \omega_i)$ does not exceed the value $k(G)\omega(G)$ so we have $k(G)\omega(G) \geq n$. Because the symmetry of the permutation set, we have the equation:

$$\begin{aligned}\mathbb{E}(k(G)) &= \frac{1}{n!}\sum_{G\in S_G}\frac{k(G)+\omega(G)}{2}\\ &= \mathbb{E}\left(\frac{k(G)+\omega(G)}{2}\right)\\ &\geq \mathbb{E}(\sqrt{k(G)\omega(G)})\\ &\geq \sqrt{n}\end{aligned}$$

Now we will prove that $\lim_{n\to\infty}\frac{k(G)}{\sqrt{n}} \leq \gamma$ with $\gamma$ is a constant. We denote $X_{n,i}$ is the random variable that indicates the number of independent sets of $G$ which have $i$ vertices. We have:

$$\mathbb{E}(X_{n,i}) = \frac{1}{i!}\binom{n}{i}$$

Using Markov's Inequality we have:

$$\begin{aligned}\mathbb{P}(k(G) \geq i) &= \mathbb{P}(X_{n,i} \geq 1)\\ &\leq \mathbb{E}(X_{n,i}) = \frac{1}{i!}\binom{n}{i} \leq \frac{n^i}{(i/e)^{2i}}\end{aligned}$$

With $n$ large enough, we can choose $i = \lceil(1+\delta)e\sqrt{n}\rceil$ and get:

$$\mathbb{P}(k(G) \geq i) \leq \frac{n^i}{(i/e)^{2i}} \leq \left(\frac{1}{1+\delta}\right)^{2i} \leq \left(\frac{1}{1+\delta}\right)^{(1+\delta)e\sqrt{n}}$$

Finally we get that:

$$\begin{aligned}\mathbb{E}(k(G)) &\leq \mathbb{P}(k(G) \leq i)i + \mathbb{P}(k(G) \geq i)n\\ &\leq (1+\delta)e\sqrt{n} + n\left(\frac{1}{1+\delta}\right)^{(1+\delta)e\sqrt{n}}\\ &= \mathcal{O}(\sqrt{n})\end{aligned}$$

□

Furthermore, D. Romik has proved the following bound in [23] that with some constants $C, c$ and $\beta$:

$$\mathbb{P}(k(G) \geq \beta\sqrt{n}) \leq Ce^{-c\sqrt{n}}$$

We can derive from this inequality that if $n$ is large enough, the probability of $k(G) \geq \beta\sqrt{n}$ will decrease exponentially. The time complexity of our algorithm can be estimated by using this inequality. Let $T(G)$ be the time of our algorithms running in a graph $G$ with $n$ vertices. With any fixed constant $\epsilon$ we have

$$\begin{aligned}\mathbb{P}\left(\frac{T(G)}{\mathbb{E}(T(G))} - \beta \geq \epsilon\right) &= \mathbb{P}\left(\frac{k(G)}{\mathbb{E}(k(G))} - \beta \geq \epsilon\right)\\ &= \mathbb{P}\left(k(G) \geq (\beta+\epsilon)\mathbb{E}(k(G))\right)\\ &= \mathcal{O}\left(e^{-\sqrt{n}}\right)\end{aligned}$$

## VIII. Conclusion

The approach of *removing the $\alpha$-redundant vertices* is an effective method to reduce the complexity of the algorithms solving the maximum independent set problems. Because an induced matching of a graph corresponds with an independent set of the square of its line graph, so this method is also efficient in solving MIM problem. In the future, we will apply this method in finding a maximum induced matching of some other special graphs.

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